A characterization of nearest neighbor intrinsic autoregressions

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Intrinsic autoregressions have been introduced in Künsch (1987) as the spatial analogue of integrated autoregressive time series models. However, their definition is more complicated than in the time series case. We discuss here in particular the nearest neighbor case. Then the intrinsic autogregression $(Z(x))$ on the infinite lattice $\mathbb{Z}^2$ is defined by its spectrum

$$f(\omega) = \sigma^2(1 - \beta \cos \omega_1 - (1-\beta)\cos \omega_2)^{-1} \quad (\omega \in (-\pi, \pi]^2) \tag{1}$$

where $\sigma^2 > 0$ and $0 < \beta < 1$. This spectral density is not integrable at the origin which expresses the fact that the model is not stationary. But covariances of increments can be computed in the usual way

$$\mathrm{Cov}(\sum_x \lambda(x) Z(x),\ \sum_x \nu(x) Z(x)) = (2\pi)^{-2} \int_{(-\pi,\pi]^2} \sum_x \lambda(x) e^{i\omega x} \sum_x \nu(x) e^{-i\omega x} f(\omega) d\omega. \tag{2}$$

Here an increment is a finite linear combination with coefficients $\lambda(x)$ summing up to zero, that is $\sum \lambda(x) = 0$. Then $\sum \lambda(x) e^{i\omega x}$ is zero at $\omega = 0$ and the integral exists.

Such an intrinsic autoregression satisfies

$$E[Z(x)|Z(x'), x' \neq x] = \frac{\beta}{2}(Z(x+e_1) + Z(x-e_1)) + \frac{(1-\beta)}{2}(Z(x+e_2) + Z(x-e_2)), \tag{3}$$

where $e_1 = (1,0)^T$ and $e_2 = (0,1)^T$, if we interprete the left-hand side of (3) as the best intrinsic predictor, see Künsch (1987, Theorem 2.2). Furthermore

$$E[(Z(x) - E[Z(x)|Z(x'), x' \neq x])^2] = \sigma^2. \tag{4}$$

Properties (3) and (4) are certainly more intuitive than the definition (1). Still there is a big difference between the time series and the spatial case. A first order integrated autoregressive process is simply a random walk, i.e. first differences are uncorrelated. In contrast to this, the increments of the nearest-neighbor intrinsic autoregression $(Z(x))$,

$$\Delta_j Z(x) = Z(x+e_j) - Z(x) \quad (j = 1, 2), \tag{5}$$

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have by formula (2) complicated auto- and crosscorrelations:

$$\mathrm{Cov}(\Delta_j Z(x+x'), \Delta_k Z(x')) = \int e^{i\omega x} g_{jk}(\omega) d\omega \tag{6}$$

where

$$g_{jk}(\omega) = (e^{i\omega_j} - 1)(e^{-i\omega_k} - 1) f(\omega). \tag{7}$$

The matrix $g(\omega) = (g_{jk}(\omega))$ is called the spectral matrix of $(\Delta_1 Z, \Delta_2 Z)^T$.

However in two dimensions, increments have to be dependent because their sum over closed loops must be zero:

$$\Delta_1 Z(x) + \Delta_2 Z(x+e_1) - \Delta_1 Z(x+e_2) - \Delta_2 Z(x) \equiv 0.$$

The following result shows that apart from these constraints the increments are all uncorrelated. Thus the difference between the time series and the spatial case is due only to the more complicated geometry. This is additional support for the claim that integrated autoregressions are the right generalization of integrated autoregressive time series models. The result has been suggested to me by Julian Besag. It was announced in the discussion of Besag and Higdon (1999).

Theorem Let $(Y_1(x))$ and $(Y_2(x))$ be two uncorrelated Gaussian white noises with variances σ_1^2 and σ_2^2 respectively. Then conditionally on

$$S(x) = Y_1(x) + Y_2(x+e_1) - Y_1(x+e_2) - Y_2(x) \equiv 0,$$

$(Y_1(x), Y_2(x))^T$ has the same auto- and crosscorrelations as the increments of a nearest neighbor intrinsic autoregression with

$$\sigma^2 = \sigma_1^2 \sigma_2^2 / (2\sigma_1^2 + 2\sigma_2^2) \text{ and } \beta = \sigma_2^2/(\sigma_1^2 + \sigma_2^2).$$

Proof The proof uses the spectral theory of multivariate stationary random fields. It is completely analogous to the theory for stationary processes. The results we quote are for the latter case, but could easily be extended to the former.

First we compute the spectral matrix $(h_{ij}(\omega))$ of $(Y_1, Y_2, S)^T$. Because there are 3 by 2 matrices g_u such that

$$(Y_1(x), Y_2(x), S(x))^T = \sum_u g_u (Y_1(x-u), Y_2(x-u))^T$$

and because $(Y_1, Y_2)^T$ has a diagonal spectral matrix with diagonal elements σ_1^2 and σ_2^2, we obtain from formula (9.2.14) of Priestley (1981) that $(h_{ij}(\omega))$ is equal to

$$\begin{pmatrix} \sigma_1^2 & 0 & \sigma_1^2(1-e^{-i\omega_2}) \\ 0 & \sigma_2^2 & -\sigma_2^2(1-e^{-i\omega_1}) \\ \sigma_1^2(1-e^{i\omega_2}) & -\sigma_2^2(1-e^{i\omega_1}) & \sigma_1^2|1-e^{i\omega_2}|^2 + \sigma_2^2|1-e^{i\omega_1}|^2 \end{pmatrix}.$$

Therefore by formulae (10.3.6) and (9.2.33) of Priestley (1981), the best predictions of (Y_1) and (Y_2) given (S) are

$$(\widehat{Y}_1(x), \widehat{Y}_2(x))^T = \sum_u c_u S(x-u)$$

where

$$\sum_u c_u e^{-i\omega u} = \begin{pmatrix} h_{13}(\omega) \\ h_{23}(\omega) \end{pmatrix} h_{33}(\omega)^{-1}.$$

Furthermore, by (10.3.8) and (9.2.35) of Priestley (1981), the residual process $(Y_1(x) - \widehat{Y}_1(x), Y_2 - \widehat{Y}_2(x))$ has the spectral matrix

$$\begin{aligned}
&\begin{pmatrix} h_{11}(\omega) & h_{12}(\omega) \\ h_{21}(\omega) & h_{22}(\omega) \end{pmatrix} - \begin{pmatrix} h_{13}(\omega) \\ h_{23}(\omega) \end{pmatrix} h_{33}(\omega)^{-1} \left(h_{31}(\omega), h_{32}(\omega)\right) \\
&= \begin{pmatrix} \sigma_1^2 - \sigma_1^4 |1 - e^{-i\omega_2}|^2 h_{33}(\omega)^{-1} & \sigma_1^2\sigma_2^2(1 - e^{i\omega_1})(1 - e^{-i\omega_2}) h_{33}(\omega)^{-1} \\ \sigma_1^2\sigma_2^2(1 - e^{-i\omega_1})(1 - e^{i\omega_2}) h_{33}(\omega)^{-1} & \sigma_2^2 - \sigma_2^4 |1 - e^{i\omega_2}|^2 h_{33}(\omega)^{-1} \end{pmatrix} \\
&= \sigma_1^2\sigma_2^2 h_{33}(\omega)^{-1} \begin{pmatrix} |1 - e^{i\omega_1}|^2 & (1 - e^{-i\omega_1})(1 - e^{i\omega_2}) \\ (1 - e^{i\omega_1})(1 - e^{-i\omega_2}) & |1 - e^{i\omega_2}|^2 \end{pmatrix}.
\end{aligned}$$

This is the same matrix as (7) if we set $\beta = \sigma_2^2/(\sigma_1^2 + \sigma_2^2)$ and $\sigma^2 = \sigma_1^2\sigma_2^2/(2\sigma_1^2 + 2\sigma_2^2)$ because $h_{33}(\omega) = 2\sigma_1^2 + 2\sigma_2^2 - 2\sigma_1^2 \cos\omega_2 - 2\sigma_2^2 \cos\omega_1$.

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References:

Besag, J. and Higdon, D. (1999) Bayesian analysis of agricultural field experiments, *J. Royal Statist. Soc. B* **61**, in press.

Künsch, H.R. (1987). Intrinsic autoregressions and related models on the two-dimensional lattice, *Biometrika*, **74**, 517–24.

Priestley, M.B. (1981). *Spectral Analysis and Time Series* Vol. 2. Multivariate Series, Prediction and Control. Academic Press, London.