A note on causal solutions for locally stationary AR-processes

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June 1995

In a series of papers, Dahlhaus (1995) has studied processes which are defined implicitly by

$$X_{t,T} = \sum_{j=1}^{p} a_j(\frac{t}{T}) X_{t-j,T} + \varepsilon_t \tag{1}$$

with ε_t i.i.d. , $E[\varepsilon_t] = 0,\ E[|\varepsilon_t|] < \infty$.

We want to show that under some conditions (1) has a sequence of solutions $(X_{t,T})$ of the form

$$X_{t,T} = \sum_{\ell=0}^{\infty} \psi_{t,T,\ell}\ \varepsilon_{t-\ell} \tag{2}$$

with

$$\sup_{t,T} \sum_{\ell=0}^{\infty} |\psi_{t,T,\ell}| < \infty. \tag{3}$$

(This implies a.s. convergence of the series in (2)).

We introduce the matrix

$$A(u) = \begin{pmatrix} a_1(u) & a_2(u) & \dots & \dots & a_p(u) \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

and put $A(u) = A(0)$ for $u < 0$.

We require the following two conditions for $(a_j(u))$:

(C1) $a_j(u)$ is continuous on $[0, 1]$ for all j.

(C2) There exists a $\delta > 0$ such that for all $|z| \leq 1 + \delta$ and for all u

$$1 - \sum_{j=1}^{p} a_j(u) z^j \neq 0.$$

It is well known that (C2) is equivalent to

$$|\lambda_j(u)| < \frac{1}{1+\delta} \quad \forall j,\ \forall u \tag{4}$$

where $\lambda_j(u)$ are the eigenvalues of $A(u)$.

It is also clear that for each T and $t = 1, \ldots, T$

$$X_{t,T} = \sum_{\ell=0}^{\infty} \left(\prod_{k=0}^{\ell-1} A(\frac{t-k}{T}) \right)_{11} \varepsilon_{t-\ell} \tag{5}$$

is a solution of (1). (Because (4) holds for $u = 0$ and we set $A(u) = A(0)$ for $u \leq 0$, the sum on the right-hand side converges and it is straightforward to check that (5) satisfies (1)). Hence our task is to show that

$$\sup_{t,T} \sum_{\ell=0}^{\infty} | \left(\prod_{k=0}^{\ell-1} A(\frac{t-k}{T}) \right)_{11} | < \infty. \tag{6}$$

We show that a stronger property holds:

$$\sup_{t,T} | \left(\prod_{k=0}^{\ell-1} A(\frac{t-k}{T}) \right)_{11} | \leq \text{const.}\ \rho^{\ell} \tag{7}$$

for some $\rho < 1$.

The proof is based on the following well known result (see e.g. Householder (1964), p. 46).

Lemma: If A is a matrix whose eigenvalues are all bounded in absolute value by c and $\varepsilon > 0$ is given, then there exists another matrix M such that

$$||A||_M \leq c + \varepsilon,$$

where $||A||_M = \sup\{||Ax||_M \ ;\ ||x||_M \leq 1\}$ and $||x||_M = ||M^{-1}x||_1 = \sum_{i=1}^{p} |(M^{-1}x)_i|$.

From (C1) and (4) we see that to any $u \in [0,1]$ there exists a neighborhood $I(u)$ such that

$$||A(v)||_{M(u)} \ \leq\ \rho < 1 \text{ if } v \in I(u),\ u \in [0,1].$$

($M(u)$ is the matrix according to the lemma such that $||A(u)||_{M(u)} \ \leq\ (1+\frac{\delta}{2})^{-1}$ and ρ is suitably chosen, e.g. $\rho = (1+\frac{\delta}{3})^{-1}$.)

Because of compactness there are finitely many u's, say $u_1, u_2, \ldots u_n$ such that $I(u_1) \cup I(u_2) \cup \ldots I(u_n) \supseteq [0,1)$. Because of $A(u) = A(0)$ for $u \leq 0$ this union even covers $(-\infty, 1]$. We assume that the $I(u_i)$ are disjoint intervals.

Now we choose a constant such that for any B

$$||B||_1 \leq \text{ const. } ||B||_{M(u_i)}$$

for $i = 1, \ldots, n$. Next fix t and T and denote by J_i the set $\{k \geq 0 \ ;\ \frac{t-k}{T} \in I(u_i)\}$ and $J_{i,\ell} = J_i \cap \{0, 1, \ldots, \ell-1\}$. Then

$$
\begin{aligned}
|\left(\prod_{k=0}^{\ell-1} A(\frac{t-k}{T})\right)_{11}| &\leq ||\prod_{k=0}^{\ell-1} A(\frac{t-k}{T})||_1 \leq \prod_{i=1}^{n} ||\prod_{k\in J_{i,\ell}} A(\frac{t-k}{T}||_1 \\
&\leq \text{const.}^n \prod_{i=1}^{n} ||\prod_{k\in J_{i,\ell}} A(\frac{t-k}{T})||_{M(i)} \\
&\leq \text{const.} \prod_{i=1}^{n} \rho^{|J_{i,\ell}|} = \text{const.}^n \rho^{\ell}.
\end{aligned}
$$

This completes the proof of (7). (Note that n is fixed.)

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